

Observables of the generalized 2D Yang–Mills theories on arbitrary surfaces: a path integral approach

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Abstract

Using the path integral method, we calculate the partition function and the generating functional (of the field strengths) of the generalized 2D Yang–Mills theories in the Schwinger–Fock gauge. Our calculation is done for arbitrary 2D orientable, and also nonorientable surfaces.

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1 Introduction

The 2D Yang–Mills theory (YM₂) is a theoretical laboratory for understanding the main theory of particle physics, QCD₄. It is well known that YM₂ is a solvable model, and in recent years there have been much effort to analyse the different aspects of this theory. A lattice formulation of YM₂ has been known for a long time [1], and the physical quantities, like the partition function and the expectation values of the Wilson loops, have been calculated in this context for arbitrary compact Riemann surfaces [2, 3]. The continuum (path integral) approach has also been studied in [4] and [5], and the Green functions of field strengths have been calculated in [6-8].

In recent years, an interesting generalization of the 2D Yang–Mills theories has been introduced. In fact, as a 2D counterpart of the theory of strong interactions, pure YM₂ is not unique, and it is possible to generalize it without losing properties such as invariance under area-preserving diffeomorphisms and lack of propagating degrees of freedom. In ref.[9], these generalized model have been introduced in the framework of the BF theory, and the partition function of these theories have been obtained by regarding the general Yang–Mills action as perturbation of the topological theory at zero area. In [10], the large N -limit of such theories (called generalized 2D Yang–Mills theories (gYM₂’s) in [10]), when coupled to fundamental fermions of $SU(N)$ have been studied. And the authors of [11] have generalized the Migdal’s suggestion about the local factor of plaquettes, and have shown that this generalization satisfies the necessary requirements. In this way they have found the partition function and the expectation values of Wilson loops of gYM₂’s. For a review see [12].

In this paper we are going to handle gYM₂ by the standard path integral approach. In section 2, we calculate the partition function, the wave functions, and the expectation values of Wilson loops of gYM₂, by doing the path integration of the corresponding BF theory in the Schwinger–Fock gauge. Our calculation is done for arbitrary orientable, and also nonorientable surfaces. It is then seen that our results are in agreement with those of [11]. In section 3, we investigate the question of the correlation functions of field strengths of gYM₂. For this purpose, we calculate the generating functional of these fields, and show that our general results reduce to the known results of YM₂ [7, 8] in the special case.

2 The wave functions of gYM₂

The gYM₂ is defined by

$$\exp S(\xi) := \int DB \exp[\mathcal{S}(B, \xi)], \quad (1)$$

where

$$\mathcal{S}(B, \xi) := \int [-i \text{tr}(B\xi) + f(B)] d\mu. \quad (2)$$

In this equation, $\xi(x)$ is defined through

$$\epsilon_{\mu\nu}\xi(x) := F_{\mu\nu}(x), \quad (3)$$

ξ and B are elements of the adjoint representation of the Lie algebra corresponding to the gauge group G , f is an arbitrary class function of B :

$$f(UBU^{-1}) = f(B), \quad \forall U \in G, \quad (4)$$

and

$$d\mu := \frac{1}{2}\epsilon_{\mu\nu}dx^\mu \wedge dx^\nu. \quad (5)$$

If $f(B)$ is a quadratic function, $f(B) = \text{tr}B^2$, it is easy to do the integration (1), and see that it is the standard YM_2 .

To begin, we calculate the wave function corresponding to (1) on a disk D with the boundary condition $\text{P exp} \oint_{\gamma=\partial D} A = g \in G$:

$$\psi_D(g) = \int D\xi DB \exp[\mathcal{S}(B, \xi)] \delta \left(\text{P exp} \oint_{\gamma} A, g \right). \quad (6)$$

One can expand the (class) delta function in terms of the characters of the irreducible representations of G :

$$\delta \left(\text{P exp} \oint_{\gamma} A, g \right) = \sum_{\lambda} \chi_{\lambda}(g^{-1}) \chi_{\lambda} \left(\text{P exp} \oint_{\gamma} A \right), \quad (7)$$

and use the following fermionic path integral representation of the Wilson loops [4, 13]:

$$\chi_{\lambda} \left(\text{P exp} \oint_{\gamma=\partial D} A \right) = \int D\eta D\bar{\eta} \exp \left[\int_0^1 \bar{\eta}(t) \dot{\eta}(t) dt + \oint_{\gamma} \bar{\eta} A_{(\lambda)} \eta \right] \eta^{\alpha}(0) \bar{\eta}_{\alpha}(1). \quad (8)$$

Here η and $\bar{\eta}$ are Grassmann valued vectors in the representation λ . In the Shwinger-Fock gauge, we also have

$$A_{\mu}^a(x) = \int_0^1 dr r x^{\nu} F_{\nu\mu}^a(rx), \quad (9)$$

and

$$F = dA. \quad (10)$$

So

$$\begin{aligned} \psi_D(g) &= \sum_{\lambda} \chi_{\lambda}(g^{-1}) \int D\eta D\bar{\eta} DB D\xi \exp \left[\int_0^1 \bar{\eta}(t) \dot{\eta}(t) dt \right] \eta^{\alpha}(0) \bar{\eta}_{\alpha}(1) \\ &\quad \times \exp \left\{ \int_D [-i \text{tr}(B\xi) + f(B) + \bar{\eta}(t) T_a(\lambda) \eta(t) \xi^a(x)] d\mu \right\}, \end{aligned} \quad (11)$$

where we have parametrized the disk by the angle (t) and radius (r) variables. $T_a(\lambda)$'s are the generators of G in the representation λ . An integration over ξ , and then B , yields

$$\psi_D(g) = \sum_{\lambda} \chi_{\lambda}(g^{-1}) \int D\eta D\bar{\eta} \exp \left\{ \int_0^1 \bar{\eta}(t) \dot{\eta}(t) dt + \int_D f[i\bar{\eta}T(\lambda)\eta] d\mu \right\} \eta^{\alpha}(0) \bar{\eta}_{\alpha}(1), \quad (12)$$

where

$$T(\lambda) := T_a(\lambda) \otimes T^a, \quad (13)$$

and

$$\bar{\eta} T(\lambda) \eta = \bar{\eta} T_a(\lambda) \eta T^a, \quad (14)$$

in which T^a 's are the generators of G in the adjoint representation. Now, using the fermionic propagator in one dimension:

$$\int D\eta D\bar{\eta} \exp \left[\int_0^1 \bar{\eta}(t) \dot{\eta}(t) dt \right] \eta^\alpha(t') \bar{\eta}_\beta(t'') = \delta_\beta^\alpha \theta(t'' - t'), \quad (15)$$

it is seen that

$$\psi_D(g) = \sum_\lambda \chi_\lambda(g^{-1}) \text{tr} \exp\{A(D)f[iT(\lambda)]\}. \quad (16)$$

Note that f can be expanded in terms of the components of B . The coefficients of this expansion are symmetric tensors, so that no ambiguity arises in (16) because of the noncommutativity of T_a 's.

Now, from (4) we have

$$f[i \, 1 \otimes U \, T(\lambda) \, 1 \otimes U^{-1}] = f[iT(\lambda)]. \quad (17)$$

However,

$$1 \otimes U \, T(\lambda) \, 1 \otimes U^{-1} = U(\lambda) \otimes 1 \, T(\lambda) \, U^{-1}(\lambda) \otimes 1, \quad (18)$$

and

$$f[i \, U(\lambda) \otimes 1 \, T(\lambda) \, U^{-1}(\lambda) \otimes 1] = U(\lambda) f[iT(\lambda)] U^{-1}(\lambda), \quad (19)$$

which shows that

$$U(\lambda) f[iT(\lambda)] U^{-1}(\lambda) = f[iT(\lambda)], \quad \forall U \in G. \quad (20)$$

This means that $f[iT(\lambda)]$ is proportional to identity:

$$f[iT(\lambda)] = f_\lambda 1_\lambda. \quad (21)$$

So, (16) becomes

$$\psi_D(g) = \sum_\lambda \chi_\lambda(g^{-1}) d_\lambda \exp[A(D)f_\lambda]. \quad (22)$$

This is the same result of [11]. Now, we proceed by the same procedure as was followed in YM_2 [4, 8] and calculate the partition function of gYM_2 on arbitrary (orientable or nonorientable) surfaces: We glue the wave functions in a suitable way (for details see [8]). The final result is

$$\psi_{\Sigma_{g,s,r}}(g_1, \dots, g_n) = \sum_\lambda h_\lambda^{r+2s} d_\lambda^{2-2g-2s-r-n} \chi_\lambda(g_1^{-1}) \cdots \chi_\lambda(g_n^{-1}) \exp[A(\Sigma_{g,s,r})f_\lambda]. \quad (23)$$

$\Sigma_{g,s,r}$ is a connected sum of an orientable surface of genus g with s Klein bottles and r projective planes, having n boundaries $\gamma_1, \dots, \gamma_n$, with boundary conditions $\text{P exp } \oint_{\gamma_i} A = g_i \in G$. h_λ is defined through

$$h_\lambda := \int \chi_\lambda(g^2) dg; \quad (24)$$

it is zero unless the representation λ is self conjugate. If so, this representation has an invariant bilinear form. Then, $f_\lambda = 1$ if this form is symmetric, and $f_\lambda = -1$ if it is antisymmetric [14].

In the case of orientable surfaces ($r = s = 0$), our general result (23) coincides with that of [11]. The expectation values of Wilson loops are also found by the same method followed in YM_2 , and leads to the same result of YM_2 , except that the second Casimir must be replaced by f_λ .

3 The generating functional $Z[J]$ of gYM_2

To calculate the Green functions of the field strength ξ^a 's, we again begin with the disk and calculate the wavefunction of gYM_2 on a disk D , with a source term coupled to ξ :

$$\psi_D[J] = \int D\xi DB \exp \left\{ \int [-i\text{tr}(B\xi) + f(B) + \xi^a J_a] d\mu \right\} \delta(\text{Pexp } \oint_{\gamma=\partial D} A, g). \quad (25)$$

Following the same steps of the previous section, we arrive at :

$$\psi_D[J] = \sum_\lambda \chi_\lambda(g^{-1}) \int D\eta D\bar{\eta} \exp \left\{ \int f[iJ^a(x) + i\bar{\eta}(t)T^a\eta(t)] d\mu \right\} \exp \left[\int_0^1 dt \bar{\eta}(t)\dot{\eta}(t) \right] \eta^\alpha(0)\bar{\eta}_\alpha(1). \quad (26)$$

Again we can expand the first exponential and calculate the resulting Green functions of the free-fermionic theory. Using the fact that the tensorial coefficients in f are totally symmetric, we finally find :

$$\psi_D[J] = \sum_\lambda \chi_\lambda(g^{-1}) \text{tr}_\lambda [\mathcal{P} \exp \int f(iJ^a(x) + iT^a) d\mu]. \quad (27)$$

In the above equation \mathcal{P} stands for ordering according to the angle variable on the disk.

As an example, consider the YM_2 , in which $f(B) = -\epsilon \text{tr}(B^2)$. In this case (27) reduces to :

$$\psi_D[J] = \exp(\epsilon \int J^a J_a d\mu) \sum_\lambda \chi_\lambda(g^{-1}) \exp[-\epsilon c_2(\lambda) A(D)] \text{tr}_\lambda [\mathcal{P} \exp 2\epsilon \int dt \int dr \sqrt{g} J(r, t)], \quad (28)$$

which is in agreement with the result obtained in [7].

Now if we glue (27) to the wavefunction (23) (with $n = 1$) :

$$Z_{\Sigma_{g,s,r}}[J] = \int dg \psi_D[J, g] \psi(\Sigma_{g,s,r}, g^{-1}), \quad (29)$$

we find the following generating functional $Z[J]$ of gYM₂ on $\Sigma_{g,s,r}$:

$$Z_{\Sigma_{g,s,r}}[J] = \sum_{\lambda} h_{\lambda}^{r+2s} d(\lambda)^{2-2g-2s-r-1} \exp [A(\Sigma_{g,s,r}) f_{\lambda}] \operatorname{tr}_{\lambda} \left\{ \mathcal{P} \exp \int f [iJ^a(x) + iT^a] d\mu \right\}. \quad (30)$$

Functional differentiating of (30) with respect to $J(x)$ gives us the Green functions of ξ 's in the Schwinger-Fock gauge.

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